Exercice 5 Soit $\ell = \lim\limits_{n \to +\infty} u_n$.

$$\forall n \in \mathbb{N}^* \qquad v_n - \ell = \frac{1}{n} \sum_{k=1}^{n} (u_k - \ell)$$

Soit $\varepsilon > 0$ $\quad \exists n_1 \quad \forall n \geqslant n_1 \quad |u_k - \ell| < \frac{\varepsilon}{2}$

$\forall n \geqslant n_1 + 1$

$$|v_n - \ell| \leq \frac{1}{n} \sum_{k=1}^{n} |u_k - \ell| \leq \frac{1}{n} \sum_{k=1}^{n_1} |u_k - \ell| + \frac{1}{n} \sum_{k=n_1+1}^{n} |u_k - \ell|$$

$$|v_n - \ell| \leq \frac{1}{n} \sum_{k=1}^{n_1} |u_k - \ell| + \frac{n - n_1}{n} \frac{\varepsilon}{2} \leq \frac{K_{n_1}}{n} + \frac{\varepsilon}{2}$$

Or, $\lim\limits_{n \to +\infty} \frac{K_{n_1}}{n} = 0$ donc

$\exists n_2 \quad \forall n \geqslant n_2 \quad \frac{K_{n_1}}{n} < \frac{\varepsilon}{2}$.

Soit $n_0 = \max(n_2, n_1 + 1)$ $\qquad \forall n \geqslant n_0 \quad |v_n - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Finalement

$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n \geqslant n_0 \quad |v_n - \ell| < \varepsilon$

i.e. $\quad \lim\limits_{n \to +\infty} v_n = \ell$

q.e.d.